MINIMAL SURFACES CONTAINING STRAIGHT LINES.*

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The Enneper-Weierstrass equations of a minimal surface† S are

$$
\begin{aligned}
x &= \frac{1}{2}\int (1 - u^2)F(u)du + \frac{1}{2}\int (1 - v^2)\phi(v)dv, \\
y &= \frac{i}{2}\int (1 + u^2)F(u)du - \frac{i}{2}\int (1 + v^2)\phi(v)dv, \qquad (1) \\
z &= \int uF(u)du + \int v\phi(v)dv.
\end{aligned}
$$

For a real surface S the functions F and ϕ are conjugate; for a real point with a real tangent plane u and v have conjugate values. The direction cosines of the normal are

$$X = \frac{u + v}{uv + 1}, \qquad Y = \frac{i(v - u)}{uv + 1}, \qquad Z = \frac{uv - 1}{uv + 1}.$$

In the following paper we determine the function F so that a real minimal surface S shall contain one or more given straight lines or portions of such lines. In the first section we use Schwarz's formulas for a minimal surface containing a given curve and having at each point of this curve a given tangent plane. In the second section a different and somewhat simpler method is used, giving the result of the first section in a different form and other results. In both these sections the results are applied to derive certain familiar theorems and several facts believed to be new. In the last section we apply the results to double minimal surfaces showing a connection between these surfaces and minimal surfaces containing straight lines.

§ 1. **Schwarz's Equations.** In 1875 H. A. Schwarz gave the equations of a minimal surface containing a given curve and having at each point of this curve a given tangent plane.‡ If the equations of the given curve are

$$x = x(t), \qquad y = y(t), \qquad z = z(t),$$

* Presented at the meeting of the American Mathematical Society, New York, April 26, 1919.

† Darboux, Leçons sur la théorié générale des surfaces, vol. 1, 2d ed., p. 344. Eisenhart, *Differential Geometry*, p. 256.

‡ Darboux, l.c., p. 449. Eisenhart, l.c., p. 266.

and the direction cosines of the normal to the surface are $X(t)$, $Y(t)$, $Z(t)$, the equations of the surface may be written

$$\bar{x} = \frac{x(u_1) + x(v_1)}{2} + \frac{i}{2}\int_{v_1}^{u_1}\left[Z(t)\frac{dy}{dt} - Y(t)\frac{dz}{dt}\right]dt, \tag{2}$$

with similar equations for $\bar{y}$ and $\bar{z}$.

We choose as the given curve a straight line parallel to the z axis, writing

$$x = a, \qquad y = b, \qquad z = t; \qquad X = \cos\varphi, \qquad Y = \sin\varphi, \qquad Z = 0,$$

where a and b are constants and φ is a real function of t. We seek to determine $F(u)$ in (1) so that the surfaces given by (1) and (2) shall be identical. Since u, v and u_1, v_1 are the parameters of the minimal curves of the surfaces (1) and (2) respectively we may assume without restriction that u is a function of u_1 alone if the surfaces are the same. A necessary condition for this identity is obtained by equating the partial derivatives with respect to u_1 of corresponding coördinates in (1) and (2). We find

$$\frac{\partial x}{\partial u_1} = -\frac{i}{2}\sin\varphi = \frac{1}{2}(1 - u^2)F(u)\frac{du}{du_1},$$

$$\frac{\partial y}{\partial u_1} = \frac{i}{2}\cos\varphi = \frac{i}{2}(1 + u^2)\cdot F(u)\frac{du}{du_1},$$

$$\frac{\partial z}{\partial u_1} = \frac{1}{2} = uF(u)\frac{du}{du_1},$$

where t is replaced by u_1 in φ. The first two equations give

$$\tan\varphi = i\frac{1 - u^2}{1 + u^2},$$

from which $u = e^{\varphi i}$. From the third equation we then have

$$F(u) = \frac{1}{2iu^2\varphi'}.$$

Now since φ is a real function of u_1, the same is true of φ', and the latter is a real function of $\varphi = -i\log u$, so that necessarily

$$F(u) = \frac{i}{u^2}R(i\log u), \tag{3}$$

where R is a real function of its argument. Substituting this value of F in (1) and at the same time

$$\phi(v) = -\frac{i}{v^2}R(-i\log v)$$

it appears that for any real function R we have, for $uv - 1 = 0$, $dx = dy = Z = 0$, so that (3) is both necessary and sufficient that the real surface S, given by (1), contain a portion of a line parallel to the z axis.

We make two applications of the result obtained. First, let us determine what real minimal surfaces applicable to surfaces of revolution or to spiral surfaces contain a portion of a line parallel to the z axis. All such surfaces are given* by taking in (1)

$$F(u) = cu^{-2+m+ni},$$

where c, m, n are constants of which the last two are real; if $n = 0$, S is applicable to a surface of revolution; if $m = 0$, S is a spiral surface. In order that a surface of this kind contain a portion of a line parallel to the z axis

$$iR(i \log u) = cu^{m+ni} = ce^{(m+ni)\log u},$$

from which it follows that $m = 0$ and c is pure imaginary. All such surfaces are then given by $F(u) = iu^{-2+ni}$, and homothetic surfaces. If $n = 0$, S is the right helicoid; if $n \neq 0$, S is a special spiral surface, which has previously been proved to contain part of the z axis.†

As a second application let us determine all real double minimal surfaces containing a portion of a line parallel to the z axis. Real double minimal surfaces are given by (1) when‡

$$F(u) = -\frac{1}{u^4}\phi\left(-\frac{1}{u}\right).$$

We have

$$-\frac{1}{u^4}\phi\left(-\frac{1}{u}\right) = \frac{i}{u^4}u^2 R\left[-i \log\left(-\frac{1}{u}\right)\right] = \frac{i}{u^2}R(i \log u + \pi).$$

It is to be noted that the expression last written has the same form as $F(u)$, which is necessarily the case since it must again give S if used in (1) in place of $F(u)$. For a double surface this expression is equal to $F(u)$; a necessary and sufficient condition is that R have the period π. If we choose (a) R constant, S is the right helicoid; (b) $R(x) = 2 \sin 2x$, $F(u) = 1/u^4 - 1$, S is Henneberg's surface;§ $R(x) = 2 \cos 2x$ gives the same surface rotated about the z axis through 45°; (c) $R(x) = -\frac{1}{2} \csc 2x$, $F(u) = 1/(1 - u^4)$, S is Scherk's surface;‡ (d) $R(x) = -4 \sin mx \cos nx$ gives

$$F(u) = \frac{(u^m - u^{-m})(u^n + u^{-n})}{u^2};$$

* Darboux, l.c., pp. 359, 368, 396.

† Transactions, vol. 19, no. 4 (1918), p. 325.

‡ Darboux, l.c., p. 410. See also p. 346.

§ Eisenhart, l.c., pp. 267, 260. Darboux, l.c., p. 327.

S is a double surface if m and n are integers whose sum is even, and is algebraic if m and n are numerically different integers.

§ 2. **Lines Parallel to the xy Plane.** We consider the condition that the real minimal surface (1) contain a line, or a portion of a line, parallel to the xy plane, given by

$$x - y \tan \beta = c_1, \qquad z = c_2,$$

using a method quite different from that of the preceding section. Along this line, supposed to lie on (1),

$$dx - \tan \beta dy = dz = \Sigma X dx = 0.$$

From these equations and (1) it follows

$$(u + v) \tan \beta + i(v - u) = 0, \qquad v = ue^{2\beta i}.$$

Substituting the value of v last given in $dz = 0$ from (1),

$$e^{-2\beta i}F(u) + e^{2\beta i}\phi(ue^{2\beta i}) = 0.$$

In the last equation we write $u = u'e^{-\beta i}$, which then becomes

$$e^{-2\beta i}F(u'e^{-\beta i}) + e^{2\beta i}\phi(u'e^{\beta i}) = 0.$$

Since the two terms of this equation are conjugate for real values of u' we have necessarily

$$e^{-2\beta i}F(u'e^{-\beta i}) = iR(u'),$$

where R is a real function. We may write $F(u)$ in the form

$$F(u) = \frac{i}{u^2} R(ue^{\beta i}).$$

It is easily proved that if $F(u)$ has the form last given the line lies in the surface (1), so that the condition for $F(u)$ is sufficient as well as necessary. In particular if the surface contains a line, or part of a line, parallel to the y axis, $\beta = 0$, $F(u) = iR(u)$; to contain a line, or part of a line, parallel to the x axis, $\beta = \pi/2$, $F(u) = iR(ui)$. The last condition may conveniently be written $F(u) = p(u) + iq(u)$, where p and q are both real functions, p being an odd function and q an even function of u. It is evident that $F(u) = iq(u)$, where q is a real even function of u is both necessary and sufficient that (1) contain lines, or parts of lines, parallel to the x and y axes. The function $F(u)$ has the form last given for the right helicoid, $q = 1/u^2$, evidently the only real minimal surface containing lines parallel to all the lines of a plane; for Enneper's surface,* q constant; for Henneberg's surface, $q = 1 + 1/u^4$; for Scherk's surface,

* Darboux, l.c., p. 373. Eisenhart, l.c., p. 269.

$q = 1/(1 + u^4)$. The last three as given are rotated 45° about the z axis from the positions in which they are usually given.

§ 3. **Double Surfaces.** It has been stated in § 1 that the condition that equations (1) give a real double minimal surface is

$$F(u) = -\frac{1}{u^4}\phi\left(-\frac{1}{u}\right),$$

where F and ϕ are conjugate functions. Darboux has given the solution of this equation* in the form

$$F(u) = \frac{1}{u^2}\left[\varphi(u) - \varphi'\left(-\frac{1}{u}\right)\right],$$

where φ and φ' are any two conjugate functions. We have solved the equation in a different form, which may be shown equivalent to Darboux's, as follows:

Let $F(u) = P(u)/u^2$ and $\phi(u) = Q(u)/u^2$, where P and Q are conjugate. The condition for a double surface becomes

$$P(u) = -Q\left(-\frac{1}{u}\right).$$

We write $P = p + iq$ and $Q = p - iq$, where p and q are two real functions of u. The condition is replaced by the two equations,

$$p(u) = -p\left(-\frac{1}{u}\right), \qquad q(u) = q\left(-\frac{1}{u}\right).$$

Replacing u by $(u + 1/u)/2 + (u - 1/u)/2$ we may regard p and q as two real functions of the two variables, $u + 1/u$ and $u - 1/u$. The conditions for a double surface become

$$p\left(u + \frac{1}{u}, u - \frac{1}{u}\right) = -p\left(-\left[u + \frac{1}{u}\right], u - \frac{1}{u}\right),$$

$$q\left(u + \frac{1}{u}, u - \frac{1}{u}\right) = q\left(-\left[u + \frac{1}{u}\right], u - \frac{1}{u}\right).$$

Evidently it is both necessary and sufficient for a real double surface that F and ϕ be conjugate and that

$$F = \frac{1}{u^2}p\left(u + \frac{1}{u}, u - \frac{1}{u}\right) + \frac{i}{u^2}q\left(u + \frac{1}{u}, u - \frac{1}{u}\right),$$

where p is an odd function and q an even function of the argument $u + 1/u$. Henneberg's surface, for example, is given in two different

* L.c., pp. 411, 412.

positions by

$$F(u) = 1 - \frac{1}{u^4} = \frac{1}{u^2}\left(u + \frac{1}{u}\right)\left(u - \frac{1}{u}\right),$$

$$F(u) = i\left(1 + \frac{1}{u^4}\right) = \frac{i}{u^2}\left[\left(u + \frac{1}{u}\right)^2 - 2\right] = \frac{i}{u^2}\left[\left(u - \frac{1}{u}\right)^2 + 2\right].$$

It is clear that in the first expression p is an odd function, in the second both values of q are even functions of $u + 1/u$. It may be noted that the generalization of Henneberg's surface given by Darboux,*

$$F(u) = \frac{1}{u^2}\left(u + \frac{1}{u}\right)^\beta\left(u - \frac{1}{u}\right)^\alpha,$$

β being an odd integer, is an immediate consequence of the preceding result. The general solution of the functional equation for real double surfaces can be given a third form, which may be convenient for some purposes, namely

$$F(u) = \frac{i}{u^2}R\left[u - \frac{1}{u},\ i\left(u + \frac{1}{u}\right)\right],$$

where R is any real function of the two arguments.

We consider again the problem of § 1, to determine $F(u)$ so that the surface given by (1), supposed real, shall contain a line, or part of a line, parallel to the z axis. We use the method of § 2. We require that for $uv - 1 = 0$, $dx = dy = 0$. From $dx = 0$, we find, substituting $v = 1/u$,

$$F(u) = -\frac{1}{u^4}\phi\left(\frac{1}{u}\right),$$

and observe that if this condition is satisfied we have $dy = 0$ along the same line. This equation resembles so closely that for a double surface that it is natural to solve it in the same way. The solution may be given the three forms,

$$F(u) = \frac{1}{u^2}\left[\varphi(u) - \varphi'\left(\frac{1}{u}\right)\right], \qquad \varphi \text{ and } \varphi' \text{ conjugate};$$

$$F(u) = \frac{1}{u^2}p\left(u + \frac{1}{u},\ u - \frac{1}{u}\right) + \frac{i}{u^2}q\left(u + \frac{1}{u},\ u - \frac{1}{u}\right),$$

p and q real functions of their arguments and odd and even functions respectively of $u - 1/u$;

$$F(u) = \frac{i}{u^2}R\left[u + \frac{1}{u},\ i\left(u - \frac{1}{u}\right)\right],$$

R any real function of its arguments. It may be proved that these

* L.c., p. 421.

results are in agreement with those of § 1. It may also be proved that all of the special forms of $F(u)$ determined in this and the preceding sections are unchanged when $F(u)$ is replaced by $-1/u^4\phi(-1/u)$.

We now prove several theorems concerning straight lines of double surfaces. Suppose first a real double minimal surface contains lines, or parts of lines, parallel to the x and y axes; then

$$F(u) = \frac{i}{u^2} q\left(u + \frac{1}{u},\, u - \frac{1}{u}\right),$$

where q is a real function of both arguments, an even function of $u + 1/u$, and an even function of u. It follows that q is an even function of $u - 1/u$, and the surface must contain a line, or part of a line, parallel to the z axis. We may state the theorem: the necessary and sufficient condition that a real minimal surface, which contains two perpendicular lines, or parts of such lines, be a double surface is that it contain also a line, or part of a line, perpendicular to the two other lines. Real double minimal surfaces containing lines, or parts of lines, parallel to the three coördinate axes are the right helicoid, $F(u) = i/u^2$; Henneberg's surface, $F(u) = i(1 + 1/u^4)$; Scherk's surface, $F(u) = i/(1 + u^4)$.

We prove the following generalization of the preceding theorem: the necessary and sufficient condition that a real minimal surface, which contains two plane geodesics, not straight lines, in perpendicular planes, be a double surface is that it contain a line, or part of a line, parallel to the intersection of the planes of the geodesics. We remark that a plane geodesic of a surface, not a straight line, is necessarily a line of curvature, and that the normals to the surface along such a curve are perpendicular to the normal to the plane of the geodesic. Suppose a real minimal surface contains two such plane geodesics in planes parallel to the xz and the yz planes; the normals to the surface along these curves are perpendicular to the y and x axes respectively, and the equations of these two lines of curvature are then $v = u$, and $v = -u$. On substituting these values in the differential equation of the lines of curvature* of (1),

$$F(u)du^2 - \phi(v)dv^2 = 0,$$

it is easily seen that F must be a real even function of u,

$$F(u) = \frac{1}{u^2} p\left(u + \frac{1}{u},\, u - \frac{1}{u}\right),$$

If the surface is a double surface p is an odd function of $u + 1\ u$, consequently also an odd function of $u - 1/u$, and the surface therefore contains a line, or part of a line, parallel to the z axis. Conversely, if p is an even

* Eisenhart, l.c., p. 257.

function of u, and an odd function of $u - 1/u$, it is also an odd function of $u + 1/u$, and (1) is a double surface. Henneberg's and Scherk's surfaces are examples of real minimal double surfaces containing two plane geodesics, not straight lines, in planes parallel to the xz and yz planes, and part of a line parallel to the z axis, when given in the forms,

$$F(u) = 1 - \frac{1}{u^4} = \frac{1}{u^2}\left(u + \frac{1}{u}\right)\left(u - \frac{1}{u}\right),$$

$$F(u) = \frac{1}{1 - u^4} = -\frac{1}{u^2}\frac{1}{\left(u + \frac{1}{u}\right)\left(u - \frac{1}{u}\right)}.$$

No real minimal surface contains a line, or part of a line, and a plane geodesic, not a straight line, in a plane perpendicular to this line. The condition that the surface (1) contain a line parallel to the y axis is $F(u) = iR(u)$; that $v = u$ be a line of curvature is $F(u) = R_1(u)$, where R and R_1 are both real functions. These conditions are evidently incompatible.

We prove finally two theorems relating to associate surfaces of a real double minimal surface. A surface associate to the real minimal surface (1) is given by the same equations when $F(u)$ and $\phi(v)$ are replaced by $e^{\alpha i}F(u)$ and $e^{-\alpha i}\phi(v)$ respectively, where α is a real constant.* In particular $\alpha = \pi$ gives a surface symmetrical to (1) with respect to the origin, $\alpha = \pi/2$ gives the adjoint surface. We prove first that no surface associate to a real double minimal surface, except the symmetrical surface, $\alpha = \pi$, is a double surface; second, that no surface associate to a real double minimal surface, except the surface itself, $\alpha = 0$, the symmetrical surface, $\alpha = \pi$, and the surfaces adjoint to these, $\alpha = \pm\pi/2$, can contain a straight line or part of a line. As previously stated the surface S_α associate to (1) is given by replacing $F(u)$ in those equations by

$$e^{\alpha i}F(u) = \frac{1}{u^2}[p\cos\alpha - q\sin\alpha + i(p\sin\alpha + q\cos\alpha)],$$

where

$$u^2F(u) = p(h, k) + iq(h, k), \qquad h = u + \frac{1}{u}, \qquad k = u - \frac{1}{u}.$$

If (1) is a double surface p and q are odd and even functions of h respectively. If S_α is also a double surface

$$p\sin\alpha = q\sin\alpha = 0, \qquad \sin\alpha = 0,$$

and $\alpha = 0$ or π.

* Darboux, l.c., p. 379. Eisenhart, l.c., p. 263.

If S_α contains a straight line, or part of a line, we may without restriction suppose the line to be parallel to the z axis. Then $p \cos \alpha - q \sin \alpha$ and $p \sin \alpha + q \cos \alpha$ are odd and even functions of k respectively. We may put this property in evidence by writing

$$p \cos \alpha - q \sin \alpha = kf(h, k^2), \qquad p \sin \alpha + q \cos \alpha = \varphi(h, k^2).$$

Remembering that p and q are odd and even functions of h respectively when (1) is a double surface, and changing the sign of h in the preceding equations, we have

$$- p \cos \alpha - q \sin \alpha = kf(- h, k^2), \qquad - p \sin \alpha + q \cos \alpha = \varphi(- h, k^2).$$

If the first two equations are solved for p and q and the values obtained substituted in the last two the resulting equations may be written

$$kf(- h, k^2) + kf(h, k^2) \cos 2\alpha = - \varphi(h, k^2) \sin 2\alpha,$$

$$\varphi(- h, k^2) - \varphi(h, k^2) \cos 2\alpha = - kf(h, k^2) \sin 2\alpha.$$

Since an even function of k and an odd function of k cannot be equal unless both vanish, we must have $\sin 2\alpha = 0$, and $\alpha = 0, \pm \pi/2$, or π. In order that the real minimal surface (1) be a double surface and contain a line, or part of a line, parallel to the z axis it is both necessary and sufficient that p be an odd function of each of its arguments, h and k, and that q be an even function of each of its arguments. If these conditions are satisfied the symmetrical surface S_π has the same properties. In order that (1) be a double surface and that the adjoint surface $S_{\pi/2}$ contain a line, or part of a line, parallel to the z axis it is both necessary and sufficient that p be an odd function of h and an even function of k and that q be an even function of h and an odd function of k. For example the surface given by (1), when

$$F(u) = \frac{1}{u^2}\left(u + \frac{1}{u}\right),$$

is a real double surface, ϕ being of course the conjugate function, whose adjoint surface contains part of a line parallel to the z axis.

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